Tight Bounds for the First Order Marcum Q-Function

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Abstract

In this paper, we develop new bounds for the first order Marcum Q-function, which are extremely tight and tighter than any of the existing bounds to the best of our knowledge. The key idea of our approach is to derive refined approximations for the 0^{th} order modified Bessel function in the integration region of the Marcum Q-function. The new bounds are very tight and can serve as an effective means in bit error rate (BER) performance analysis for non-coherent demodulation in digital communication.

Keywords: BER, Marcum Q-function, upper bounds, lower bounds, the 0^{th} order modified Bessel function of the first kind.

I. Introduction

Performance analysis is very important in digital communication. Q-function provides an effective means to analyze BER performance for coherent demodulation. However, for non-coherent demodulation, we need to use Marcum Q-function [1]–[3] rather than Q-function to analyze BER performance.

Marcum Q-function consists of two integrals, thereby requiring complicated numerical calculation. To simplify the computation, simple lower bounds and upper bounds were derived. However, the existing lower bounds and upper bounds for Marcum Q-function are not tight. In this paper, we derive lower bounds and upper bounds, which are tighter than any of the existing bounds to the best of our knowledge. We will only consider the first order Marcum Q-function since higher order Marcum Q-function can be derived by the first order Marcum Q-function.

The first order Marcum Q-function denoted by $Q_1(a,b)$ is defined by [4]

$$Q_1(a,b) = \int_b^{\infty} x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\,dx, \quad a \geq 0, b \geq 0 \tag{1}$$

where (also see Ref. [4])

$$I_0(y) = \frac{1}{\pi} \int_0^{\pi} \exp(y \cos\theta)\, d\theta \tag{2}$$

To develop bounds on $Q_1(a,b)$, our idea is to bound the 0^{th} order modified Bessel function of the first kind indicated as Bessel function $I_0(\cdot)$ in the integrand with refined functions. We want to emphasize two key points. First, we only need to use a function to approximate $I_0(\cdot)$ in the integration region of $Q_1(a,b)$, i.e., $[b,\infty)$. Second, the maximum point of the integrand can be approximated by parameter a [4]; also, the integrand is a monotonically increasing function in $x \in [0,a]$ and a monotonically decreasing function in $x \in [a,\infty)$. Hence, if $b > a$, we intend to use a simple monotonically decreasing function to bound the integrand in $[b,\infty)$; otherwise, we intend to use a simple monotonically increasing function to bound the integrand in $[0,b]$.

Previous work on bounds for the first-order Marcum Q-function has been reported in Ref. [3]–[7]. Simon and Alouini [3] derived exponential-type bounds; the derived bounds have simple expressions, thereby simplifying the BER analysis; however, the bounds are not tight. Simon [6] obtained tight exponential-type bounds by using the series representation of the Marcum Q-function; however, Simon did not give an upper bound for the case of $b < a$. Chiani [5] used a different integral expression of the Marcum Q-function and derived tight bounds. But the upper bound in the case of $b < a$ was not given in Ref. [5]. Kam and Li [7], [8] regarded the Marcum Q-function as the probability of 2D normalized Gaussian random variables in the region outside a disc and derived tight bounds for medium values of parameters a and b; however, when a and b are both large or small, the bounds are not tight. Zhao et al. [9] proposed tight upper bound in the case $b \geq a$ based on the same geometric interpretation of the Marcum Q-function. Corazza and Ferrari [4] derived bounds that are the tightest overall in the literature, for all values of a and b; however, in the case of $b \geq a$, the smaller a becomes, the looser the lower bound becomes.

The methods proposed in this paper are intended to address all the aforementioned limitations. The new bounds, which we obtain, are extremely tight and tighter than any of the existing bounds to the best of our knowledge. By bounding $I_0(x)$ with refined functions, we overcome the weakness in Ref. [4] and thus derive extremely tight bounds for the Marcum Q-function.

The remainder of this paper is organized as follows. Section II presents our lower bounds for the first-order Marcum Q-function. In Section III, we describe our upper bounds for the first-order Marcum Q-function. Section IV compares our bounds with the existing bounds by both theoretical analysis and numerical results. We conclude the paper in Section V.

II. Lower Bounds

In this section, we present our lower bounds. Similar to Ref. [4], our idea is to bound the integrand in the integration domain $[b, \infty)$ instead of the whole support $[0, \infty)$. Note that the maximum point of the integrand can be approximated by parameter a; also, the integrand is a monotonically increasing function in $x \in [0, a]$ and a monotonically decreasing function in $x \in [a, \infty)$. Hence, if $b > a$, we aim at finding tight lower bound for function $I_0(\cdot)$ in $[b, \infty)$; otherwise, we aim at finding tight upper bound for function $I_0(\cdot)$ in $[0, b]$ to bound $1 - Q_1(a, b)$.

The following proposition shows our lower bounds for the two cases, i.e., $b \geq a$ and $b \leq a$.

Proposition 1: In the case $b \geq a \geq 0$, our lower bound, denoted as LB1-Ours, is given by

$$\text{LB1-Ours} \triangleq \sqrt{\frac{\pi}{8}} \frac{b I_0(ab)}{\sinh(ab)} \left[\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right] \tag{3}$$

In the case $a \geq b \geq 0$, our lower bound, denoted as LB2-Ours, is given by

$$\text{LB2-Ours} \triangleq 1 - \sqrt{\frac{\pi}{2}} \frac{b I_0(ab)}{\sinh(ab)} \left[\operatorname{erf}\left(\frac{a}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{a-b}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{a+b}{\sqrt{2}}\right) \right] \tag{4}$$

Proof: Here we take into account the asymptotic form for the Bessel function $I_0(x)$; the following asymptotic expression is well known [10]

$$I_0(x) \to \frac{1}{\sqrt{2\pi x}} e^x, \quad x \gg \frac{1}{4} \tag{5}$$

Define

$$f(x) \triangleq \frac{1}{\sqrt{2\pi x}} e^x$$

and consider function $h(x)$, defined by

$$h(x) = \frac{\sinh x}{x}, \quad x > 0$$

It is easy to see that $\frac{f(x)}{h(x)}$ is monotonically increasing in $x > 1$ since

$$
\begin{aligned}
\left(\frac{f(x)}{h(x)}\right)' &= \frac{e^{2x} - 1 - 4x}{4\sqrt{2\pi x}\sinh^2 x} \\
&> \frac{1 + 2x + \frac{4x^2}{2!} - 1 - 4x}{4\sqrt{2\pi x}\sinh^2 x} \\
&> \frac{2x^2 - 2x}{4\sqrt{2\pi x}\sinh^2 x} \\
&> 0, \qquad \text{for } x > 1
\end{aligned}
\tag{6}
$$

The Bessel function and its asymptotic function are shown in Fig. 1 where we can see they are equal when $x \approx 0.26$ and approach each other very fast when $x > 0.26$. The inequality in (6) shows that $f(x)$, i.e., the asymptotic function for Bessel function, is growing faster than $h(x)$. With the asymptotic property in Eq. (5), it assures that $I_0(x)$ should grow faster than $h(x)$ when x gets large enough. For small value of x, however, we cannot use this asymptotic characteristic any more and thus resort to numerical simulation which is not mathematically strict but still can clarify its accuracy. As a result, we examine numerical value of the derivative function, given by

$$
\begin{aligned}
\left(\frac{I_0(x)}{h(x)}\right)' &= \left(\frac{xI_0(x)}{\sinh x}\right)' \\
&= \frac{[I_0(x) + xI_1(x)]\sinh x - xI_0(x)\cosh x}{\sinh^2 x}
\end{aligned}
$$

Since the denominator is positive we just consider the numerator and examine its sign. Define

$$
g(x) \triangleq [I_0(x) + xI_1(x)]\sinh x - xI_0(x)\cosh x
$$

and its numerical plot is presented in Fig. 2 from which we can clearly know that $g(x)$ is increasing exponentially from the origin. Actually we have tested large values of x to find that $g(x)$ is exponentially increasing with x. Meanwhile, when x is large enough, e.g., $x > 50$, the Bessel function could be replaced by its asymptotic function $f(x)$ defined in Eq. (5) already for analysis.

Combing the asymptotic approximation for larger x and numerical illustration for smaller x, we safely say that $\frac{I_0(x)}{h(x)}$ is also increasing in $x > 0$ just as $\frac{f(x)}{h(x)}$ defined in Eq. (6) and thus the following inequalities can be verified.

$$I_0(x) \geq \frac{bI_0(b)}{\sinh b} \cdot \frac{\sinh x}{x}, \qquad x \geq b \geq 0 \tag{7}$$

$$I_0(x) \leq \frac{bI_0(b)}{\sinh b} \cdot \frac{\sinh x}{x}, \qquad 0 < x \leq b \tag{8}$$

The functions involved in inequalities Eq. (7) and Eq. (8) are shown in Fig. 3 for $b = 4$.

Using the above two inequalities, we can derive two lower bounds in the cases $b \geq a \geq 0$ and $a \geq b \geq 0$. Note that in the case $a \geq b \geq 0$, we can bound $1 - Q_1(a, b)$ instead of $Q_1(a, b)$ since the integrand is monotonically increasing in the integration region of $1 - Q_1(a, b)$.

Case $b \geq a \geq 0$:

$$\begin{aligned} Q_1(a,b) &= \int_b^\infty x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\, dx \\ &\geq \frac{bI_0(ab)}{\sinh ab} \int_b^\infty \exp\left(-\frac{x^2+a^2}{2}\right) \sinh(ax)\, dx \\ &= \sqrt{\frac{\pi}{8}} \frac{bI_0(ab)}{\sinh(ab)} \left[\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right] \\ &\triangleq \text{LB1-Ours} \end{aligned} \tag{9}$$

Thus, Eq. (9) proves that Eq. (3) is indeed a lower bound for the Marcum Q-function in the case $b \geq a \geq 0$.

Case $a \geq b \geq 0$:

$$
\begin{aligned}
Q_1(a,b) &\geq 1 - \frac{bI_0(ab)}{\sinh(ab)} \int_0^b \exp\left(-\frac{x^2+a^2}{2}\right) \sinh(ax)\, dx \\
&= 1 - \sqrt{\frac{\pi}{2}} \frac{bI_0(ab)}{\sinh(ab)} \left[\operatorname{erf}\left(\frac{a}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{a-b}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{a+b}{\sqrt{2}}\right)\right] \\
&\triangleq \text{LB2-Ours} \qquad (10)
\end{aligned}
$$

Combining Eq. (9) and Eq. (10), we complete the proof of Eq. (3) and Eq. (4). ■

In Section IV, we will show that our lower bounds are extremely tight.

III. Upper Bounds

Similar to deriving the lower bounds, in order to obtain upper bounds for $Q_1(a,b)$, we aim at finding tight upper bound for the function $I_0(\cdot)$ in $[b,\infty)$ if $b > a$; otherwise, we aim at finding tight lower bound for the function $I_0(\cdot)$ in $[0,b]$ to bound $1 - Q_1(a,b)$.

The following proposition shows our upper bounds for the two cases, i.e., $b \geq a$ and $b \leq a$.

Proposition 2: In the case $b \geq a \geq 0$, our upper bound, denoted as UB1-Ours, is give by

$$
\begin{aligned}
&\text{UB1-Ours} \\
&\triangleq \frac{I_0(ab)+3}{e^{ab}+3} \left\{\exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + 3\exp\left(-\frac{a^2+b^2}{2}\right)\right\} \qquad (11)
\end{aligned}
$$

In the case $a \geq b \geq 0$, our upper bound, denoted as UB2-Ours, is given by

$$
\begin{aligned}
&\text{UB2-Ours} \\
&\triangleq 1 - \frac{I_0(ab)}{e^{ab}+3} \left\{4\exp\left(-\frac{a^2}{2}\right) - \exp\left[-\frac{(b-a)^2}{2}\right]\right. \\
&\quad \left. - 3\exp\left(-\frac{a^2+b^2}{2}\right) + a\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]\right\} \qquad (12)
\end{aligned}
$$

Proof: Consider function $r(x)$ defined below,

$$r(x) = \frac{I_0(x)}{e^x + m}, \quad x \geq 0, m \geq 0; \tag{13}$$

We want to choose maximized m such that $r(x)$ is monotonically decreasing. It will be shown in Section IV that the larger m, the tighter the upper bounds provided that $r(x)$ is monotonically decreasing.

Actually in Ref. [4] the authors have used $r(x)$ in the case $m = 0$ and given the following simple proof [4] to show $r(x)$ is decreasing in $x > 0$.

$$\begin{aligned} r^{'}(x) &= \left(\frac{I_0(x)}{e^x}\right)^{'} \\ &= \frac{I_1(x) - I_0(x)}{e^x} \\ &< 0 \quad \text{for } x > 0 \end{aligned}$$

In order to choose lager value of m and obtain tighter upper bounds, we consider the derivative function of $r(x)$ defined in Eq. (13), i.e.,

$$r^{'}(x) = \left(\frac{I_0(x)}{e^x + m}\right)^{'} \tag{14}$$

$$= \frac{I_1(x)(e^x + m) - I_0(x)e^x}{(e^x + m)^2} \tag{15}$$

Define

$$\begin{aligned} p(x) &\triangleq I_1(x)(e^x + m) - I_0(x)e^x \\ &= e^x \left[I_1(x)\left(1 + \frac{m}{e^x}\right) - I_0(x)\right] \end{aligned}$$

Obviously this function is increasing first and decreasing after the maximum point, as shown in Fig. 4. The maximum value of m to maintain $p(x)$ nonpositive over the whole region $x > 0$

is slightly larger than 3. For convenience, here we choose $m = 3$ and accordingly $r(x)$ becomes

$$r(x) = \frac{I_0(x)}{e^x + 3}, \quad x \geq 0 \tag{16}$$

which is monotonically decreasing in $x > 0$. Therefore, the following inequalities hold, as shown in Fig. 5.

$$I_0(x) \leq \frac{I_0(b)}{e^b + 3} \left(e^x + 3\right), \qquad x \geq b \geq 0 \tag{17}$$

$$I_0(x) \geq \frac{I_0(b)}{e^b + 3} \left(e^x + 3\right), \qquad 0 \leq x \leq b \tag{18}$$

Using the above two inequalities, we can derive two upper bounds in the cases $b \geq a \geq 0$ and $a \geq b \geq 0$.

Case $b \geq a \geq 0$:

$$\begin{aligned} & Q_1(a,b) \\ \leq & \frac{I_0(ab) + 3}{e^{ab} + 3} \int_b^{\infty} x \exp\left(-\frac{x^2 + a^2}{2}\right) \left(e^x + 3\right)\, dx \\ = & \frac{I_0(ab) + 3}{e^{ab} + 3} \left\{ \exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}}\, \text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + 3\exp\left(-\frac{a^2 + b^2}{2}\right) \right\} \\ \triangleq & \text{ UB1-Ours} \end{aligned} \tag{19}$$

With regard to upper bound for the case $a \geq b \geq 0$, we can calculate $1 - Q_1(a, b)$ instead.

Case $a \geq b \geq 0$:

$$
\begin{aligned}
& Q_1(a,b) \\
& \leq 1 - \frac{I_0(ab)}{e^{ab}+3} \int_0^b x \exp\left(-\frac{x^2+a^2}{2}\right)\left(e^{ax}+3\right)\, dx \\
& = 1 - \frac{I_0(ab)}{e^{ab}+3} \left\{ 4\exp\left(-\frac{a^2}{2}\right) - \exp\left[-\frac{(b-a)^2}{2}\right] \right. \\
& \quad \left. - 3\exp\left(-\frac{a^2+b^2}{2}\right) + a\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]\right\} \\
& \triangleq \text{UB2-Ours}
\end{aligned} \tag{20}
$$

Eq. (19) and Eq. (20) complete the proof. ■

IV. Comparison between our bounds and the existing bounds

To the best of our knowledge, Ref. [4] provides the tightest bounds in the literature. So we first focus on the comparison between our bounds and the bounds in [4] from both theoretical analysis and numerical results. In addition, the bounds in [7], [9] are also very tight and have simple expressions so we will also present the comparison results with them. To show the improvement of our bounds over the existing bounds, the numerical comparison between our bounds and other typical bounds in the literature is given at the same time.

A. *Our bounds vs. the bounds in [4]*

The following proposition shows that our bounds theoretically outperform the bounds in [4].

Proposition 3: In the defined intervals,

$$
\text{LB1-CF} \quad < \text{LB1-Ours} \leq Q_1(a,b) \tag{21}
$$

$$
\text{UB1-CF} \quad > \text{UB1-Ours} \geq Q_1(a,b) \tag{22}
$$

$$
\text{UB2-CF} \quad > \text{UB2-Ours} \geq Q_1(a,b) \tag{23}
$$

where LB1-CF is defined in [3, Eq. (9)], UB1-CF in [3, Eq. (7)], and UB2-CF in [3, Eq. (12)].

Proof: First, we prove the results for the lower bounds. Ref. [4] uses the following inequalities to derive the lower bounds for $Q_1(a,b)$.

$$I_0(x) > \frac{bI_0(b)}{e^b}\frac{e^x}{x}, \qquad x > b \geq 0, \qquad b \geq a \geq 0 \tag{24}$$

$$I_0(x) \leq \exp(\varsigma x), \qquad 0 \leq x \leq b, \qquad a \geq b \geq 0 \tag{25}$$

where

$$\varsigma = \frac{\log I_0(ab)}{b}$$

In this paper we adopt refined functions to approximate $I_0(x)$ with Eq. (7) and Eq. (8). It is easy to see that

$$\frac{bI_0(b)}{\sinh b}\frac{\sinh x}{x} > \frac{bI_0(b)}{e^b}\frac{e^x}{x}, \quad x > b \geq 0 \tag{26}$$

which means that the function for approximating $I_0(x)$ in Eq. (7) is tighter than that in Eq. (24). Therefore, our lower bound LB1-Ours is always tighter than LB1-CF. Thus, Eq. (21) holds.

Now, we prove the results for the upper bounds. Ref. [4] uses following inequalities to bound the Bessel function:

$$I_0(x) \leq \frac{I_0(b)}{e^b}\left(e^x\right), \qquad x \geq b \geq 0 \tag{27}$$

$$I_0(x) \geq \frac{I_0(b)}{e^b+3}\left(e^x\right), \qquad 0 \leq x \leq b \tag{28}$$

It is easy to prove the following inequalities, i.e.,

$$\frac{e^x}{e^b} > \frac{e^x+3}{e^b+3}, \quad x > b \tag{29}$$

$$\frac{e^x}{e^b} < \frac{e^x+3}{e^b+3}, \quad 0 < x < b \tag{30}$$

Then, from (17), (18), (27), (28), (29), and (30), we derive tighter functions to approximate the Bessel function and thus our upper bounds are tighter than the upper bounds in Ref. [4]. Therefore, Eq. (22) and Eq. (23) hold. This completes the proof. ■

We would like to emphasize that LB1-Ours outperforms LB1-CF significantly when parameter a is relatively small. Consider the case when $b \geq a$ and a is small, for example, $a = 0.1$. Then, when b is not large enough,

$$I_0(ab),\, e^{ab},\, \mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \to 1, \quad \text{when } a \to 0$$

but b is quite small, just slightly larger than a, then LB1-CF becomes quite small and loose and thus cannot bound the Marcum Q-function anymore. In contrast, LB1-Ours is still tight enough so that LB1-Ours $\leq Q_1(a, b)$ and LB1-Ours $\to 1$ since

$$\sqrt{\frac{\pi}{8}} \frac{abI_0(ab)}{\sinh(ab)} = O(1), \quad \text{when } a \to 0$$

$$\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right) = O(a), \quad \text{when } a \to 0$$

Fig. 6 indicates that when $b \geq a$ and a is small, e.g., $a = 0.5$, LB1-CF indeed becomes quite loose in the region when b is not too large, while our lower bound LB1-Ours can still tightly bound the Marcum Q-function. Moreover, with parameter a getting smaller, e.g., $a = 0.1$, LB1-Ours outperforms LB1-CF more significantly, as we can see in Fig. 7. In contrast to the fact that LB1-CF gets quite loose when a and b are small, the robustness of our lower bound LB1-Ours makes it accurate in BER estimation even under low signal to noise ratio (SNR) regime.

Eq. (22) in Proposition 3 can also be verified by numerical results. From Fig. 6 and Fig. 7, it is evident that when the parameters are small, our upper bound UB1-Ours outperforms UB1-CF obviously. With parameters getting larger, though UB1-Ours and UB1-CF tend to be equivalent, UB1-Ours is always tighter than UB1-CF as indicated by Eq. (22). UB2-Ours is slightly tighter than UB2-CF; hence, we only show the theoretical comparison in Eq. (23), without numerical comparison.

Considering LB2-Ours vs. LB2-CF, it is not easy to compare two approximated functions in Eq. (8) with Eq. (25) for $I_0(x)$. However, we can use numerical results to compare these two lower bounds for the Marcum Q-function. The comparison between LB2-Ours and LB2-CF is presented in Fig. 8, which illustrates that LB2-Ours is much tighter than LB2-CF; note that LB2-Ours has a simpler expression than LB2-CF.

B. Our bounds vs. other typical bounds

Ref. [7] proposed exponential bounds and erfc bounds. Since the erfc bounds are usually tighter than exponential bounds, we focus on the comparison between our bounds and erfc bounds UB3-KL and LB3-KL in Ref. [7], where UB3-KL is defined in [5, Eq. (18)] and LB3-KL in [5, Eq. (19)].

$$\begin{aligned}
&\text{UB3-KL} \\
&= \frac{1}{2}\operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) + \frac{1}{2}\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \\
&\quad + \frac{1}{a\sqrt{2\pi}}\left\{\exp\left[-\frac{(b-a)^2}{2}\right] - \exp\left[-\frac{(b+a)^2}{2}\right]\right\} \\
&\text{LB3-KL} \\
&= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]\cdot\left[1-\operatorname{erfc}\left(\frac{b}{\sqrt{2}}\right)\right] + \operatorname{erfc}\left(\frac{b}{\sqrt{2}}\right)
\end{aligned}$$

where UB3-KL is valid in $a > 0, b \geq 0$ and LB3-KL in $a \geq 0, b \geq 0$.

Ref. [9] obtained tight upper bound UB1-MSB defined by

$$\text{UB1-MSB} = \frac{1}{2}e^{-(b^2-a^2)/2} + \frac{\arctan(a/b)}{\pi}e^{-(b^2-a^2)/2} + \frac{\arctan(b/a)}{\pi}e^{-(b^2+a^2)/2}, b \geq a \geq 0$$

The upper bound UB1-SA, two lower bounds LB1-SA and LB2-SA in [3] are given by

$$\text{UB1-SA} = \exp\left[-\frac{(b-a)^2}{2}\right], b \geq a$$

$$\text{LB1-SA} = \exp\left[-\frac{(b+a)^2}{2}\right], b \geq a$$

$$\text{LB2-SA} = 1 - \frac{1}{2}\left\{\exp\left[-\frac{(b-a)^2}{2}\right] - \exp\left[-\frac{(b+a)^2}{2}\right]\right\}, b \leq a$$

The $I_0(\cdot)$ type tight bounds in [5] are as follows.

$$\text{UB1C} = \exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab) + a\sqrt{\frac{\pi}{8}}\,\text{erfc}\left(\frac{b-a}{\sqrt{2}}\right), b \geq a$$

$$\text{LB1C} = \exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab), b \geq a$$

$$\text{LB2C} = \exp\left(-\frac{b^2+a^2}{2}\right) I_0(ab), b \leq a$$

The numerical comparisons between our bounds and aforementioned bounds are shown in Fig. 9-13, where we show the numerical comparisons for the cases $b \geq a \geq 0$ and $a \geq b \geq 0$.

In Fig. 9, we compare the upper bounds when $a = 1$ and $b \geq a$. As can be seen in Fig. 9, our upper bound UB1-Ours outperform other bounds significantly. With increasing value of a, e.g., $a = 8$, we show the comparison result in Fig. 10 in logarithmic scale. In this case, only UB3-KL [7] has the similar performance as ours, though slightly looser than UB1-Ours.

In Fig. 11, we compare the lower bounds in the case $a = 5$ and $b \geq a$ and find our lower bound LB1-Ours is the tightest among the bounds overall. We show the numerical comparison of lower bounds for $a = 1$ and $b \leq a$ in Fig. 12 where we can see some bounds get quite loose as opposed to our lower bound LB2-Ours that is almost equal to the actual value of the Marcum Q-function.

Since only [7] gives upper bound when $b \leq a$ and the bounds in [7] are valid for $a > 0$ and $b > 0$, we consider comparing our bounds with UB3-KL/LB3-kL together. As shown in Fig. 13, our bounds outperform UB3-KL/LB3-kL overall.

As a result of the comaprisons above, we highlight the tightness and robustness of our bounds and it is worth remarking that our bounds for the first-order Marcum Q-function can potentially serve as an effective means for BER performance analysis in digital communication. The Bessel function is complicated without closed-form; in order to obtain extreme tight bounds, we resort to numerical tool, though not mathematically strict, to prove inequalities Eq. (8)(18)(7)(17) which are indeed accurate as stated in Section II and III.

V. Conclusion

This paper proposes extremely tight bounds for the first order Marcum Q-function. To the best of our knowledge, our proposed bounds outperform the tightest bounds in the literature, which are given in Ref. [4]. This is validated by theoretical analysis and numerical results. Although the bounds in Ref. [4] are quite tight in most cases, the bounds become unbounded when the argument is small. Our proposed bounds overcome this limitation; specifically, our bounds are tight no matter whether parameters a and b are large or small. Moreover, our bounds are much tighter than the other typical bounds given in [3], [5], [7], and [9]. Our bounds enjoy tightness and robustness against change of parameters a and b, and have simple expressions, resulting in reduced computation overhead. Hence, our bounds are expected to serve as a powerful tool for BER performance analysis in digital communication.

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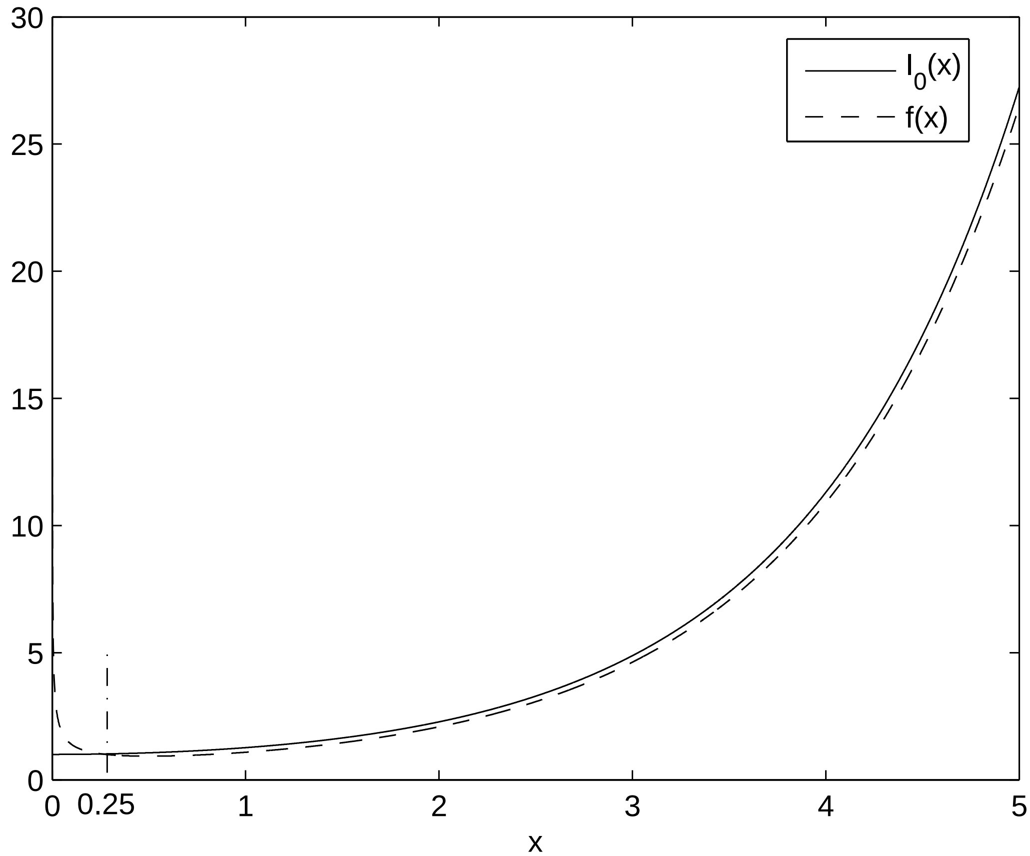

Fig. 1. The asymptotic function $f(x) \triangleq \frac{1}{\sqrt{2\pi x}} e^x$ for $I_0(x)$.

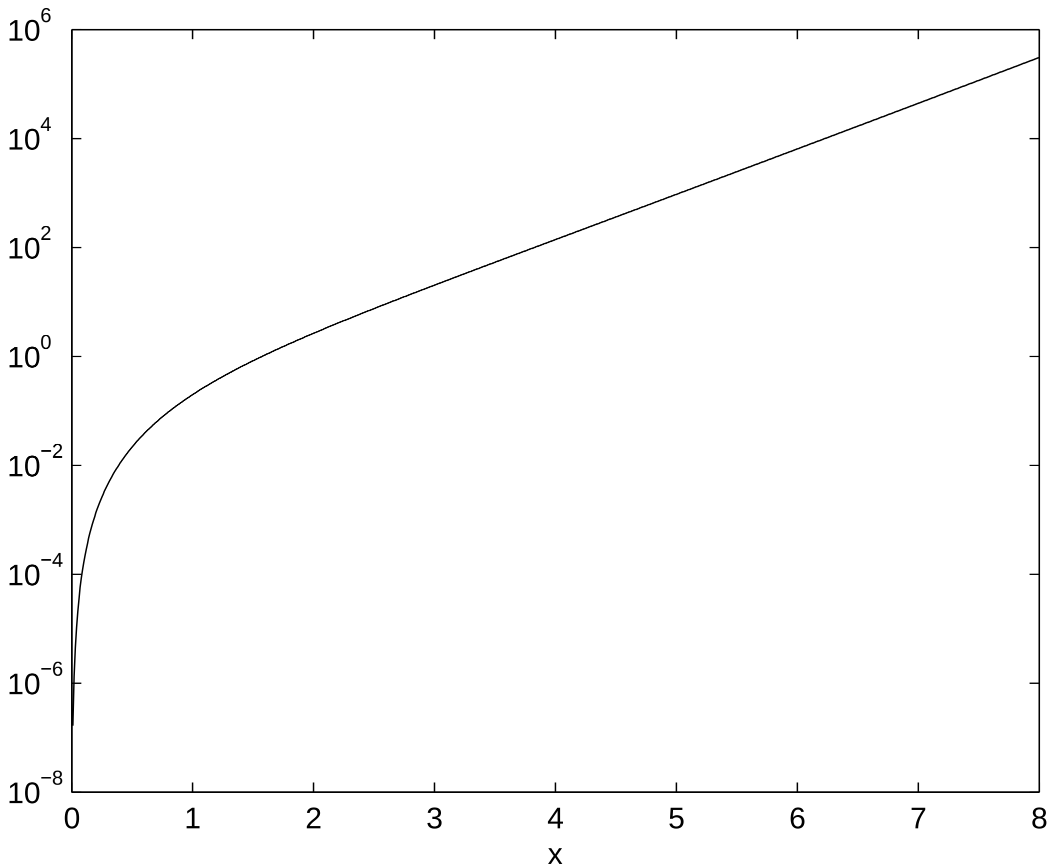

Fig. 2. The plot of $g(x) \triangleq [I_0(x) + xI_1(x)] \sinh x - xI_0(x) \cosh x$, in logarithmic scale.

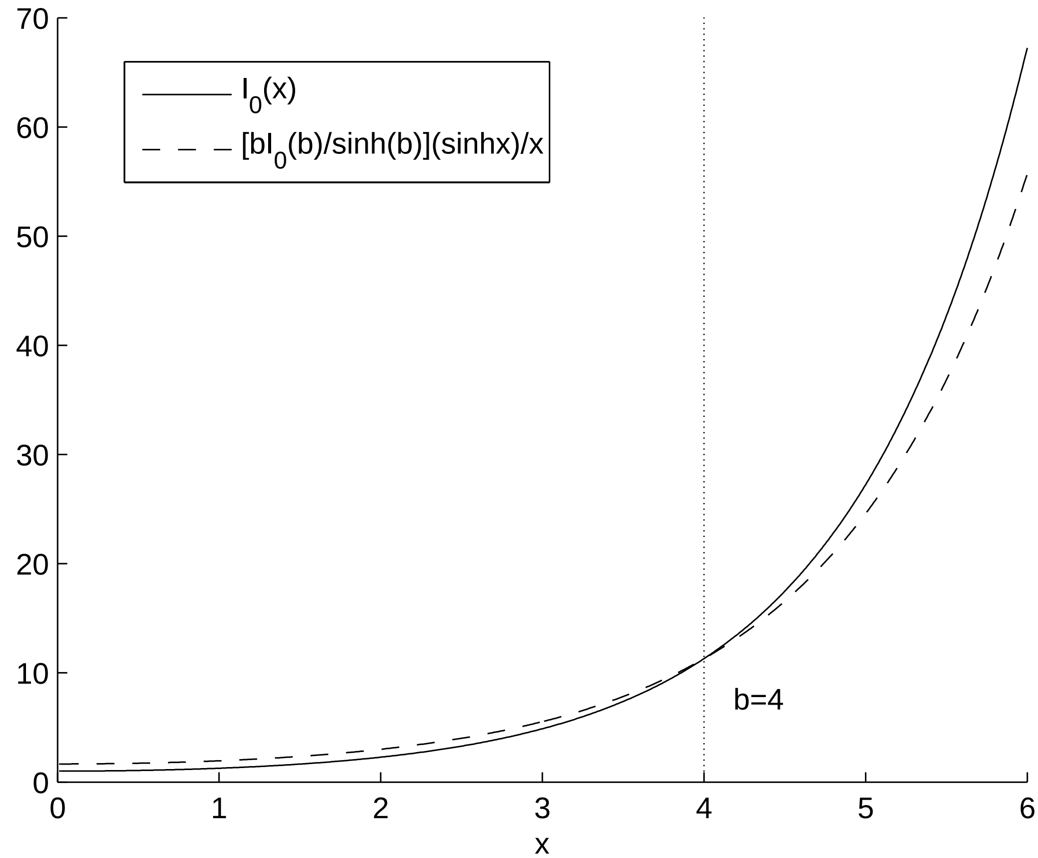

Fig. 3. Comparison between $I_0(x)$ and $\frac{bI_0(b)}{\sinh b}\frac{\sinh x}{x}$, $b=4$.

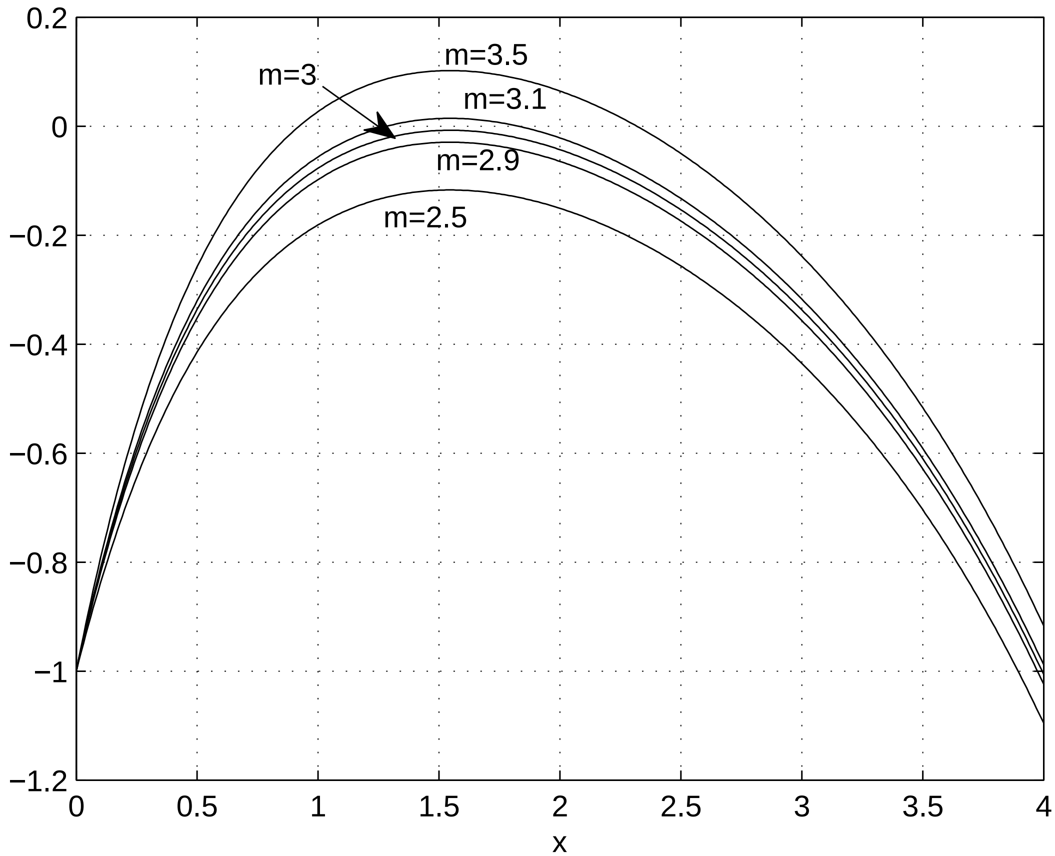

Fig. 4. Plots of $p(x) = I_1(x)(e^x + m) - I_0(x)e^x$ with $m = 2.5, 2.9, 3, 3.1, 3.5$, scaled by e^x.

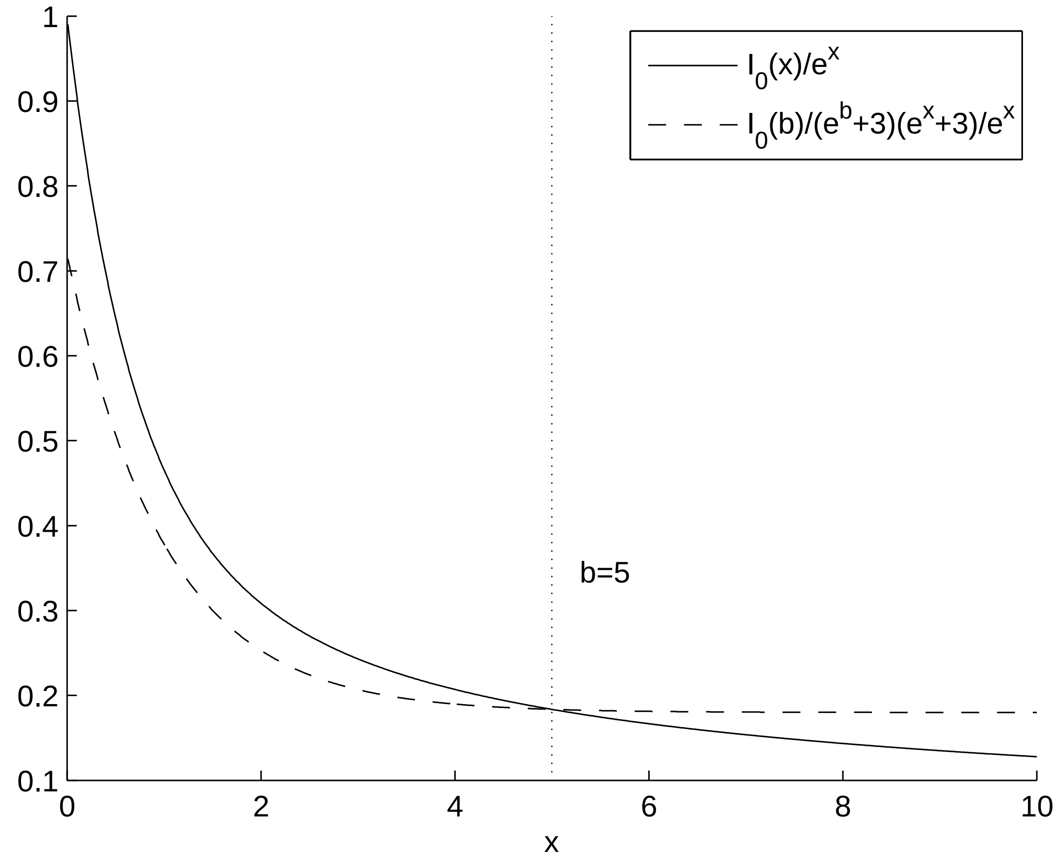

Fig. 5. Comparison between $I_0(x)$ and $\frac{I_0(b)}{e^b+3}\,(e^x+3)$ where both functions are scaled by e^x and $b=5$.

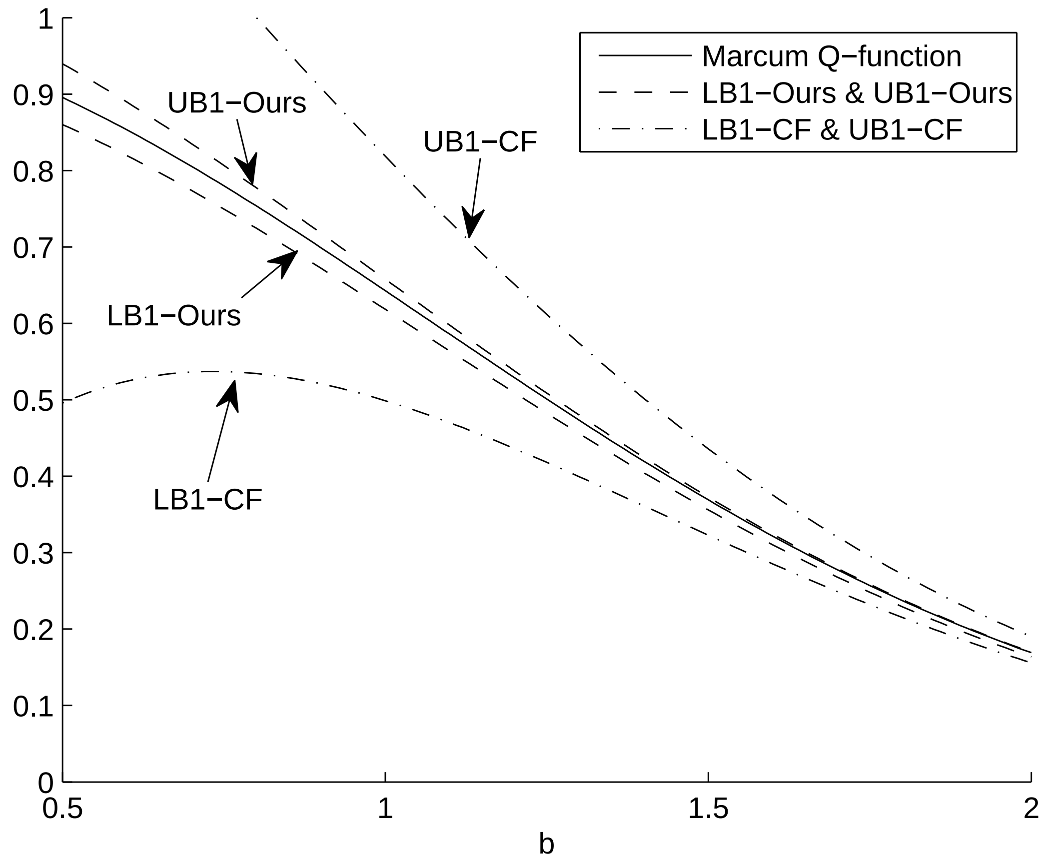

Fig. 6. The bounds comparison between our bounds and the bounds in [4], in the case $a = 0.5$ and $b \geq a$.

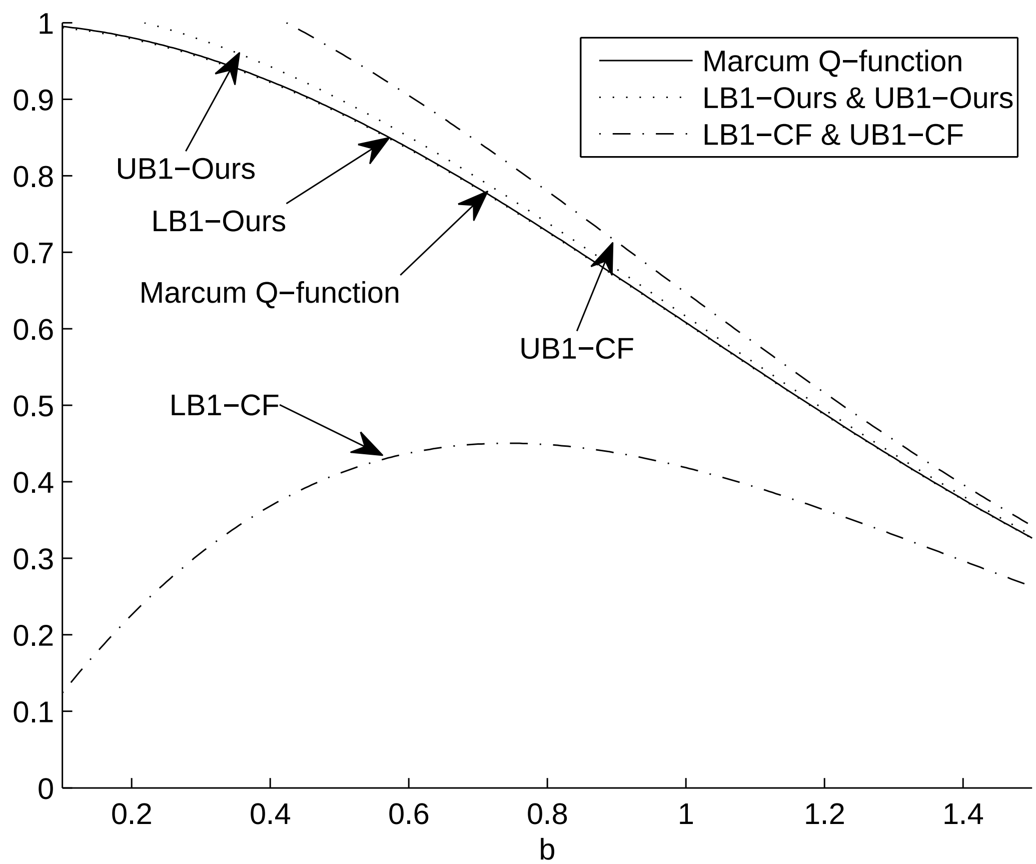

Fig. 7. The bounds comparison between our bounds and the bounds in [4], in the case $a = 0.1$ and $b \geq a$.

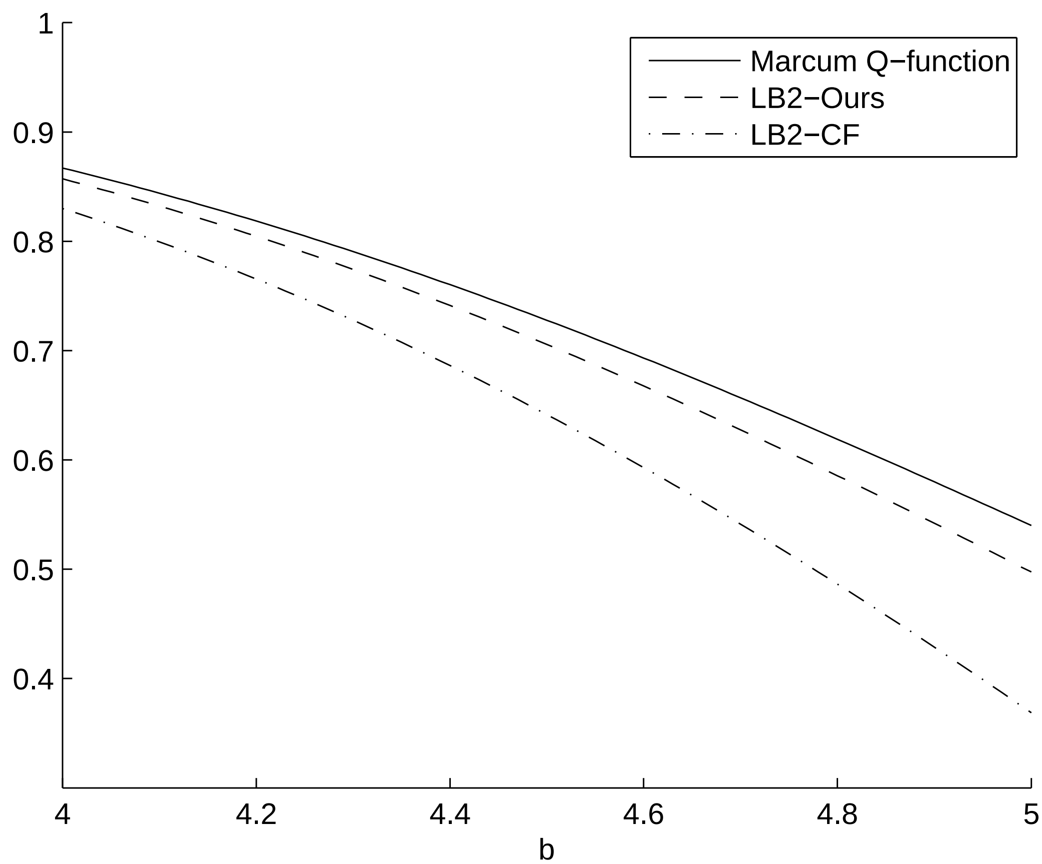

Fig. 8. The lower bounds comparison between LB2-ours and LB2-CF in [4], in the case $a = 5$ and $b \leq a$.

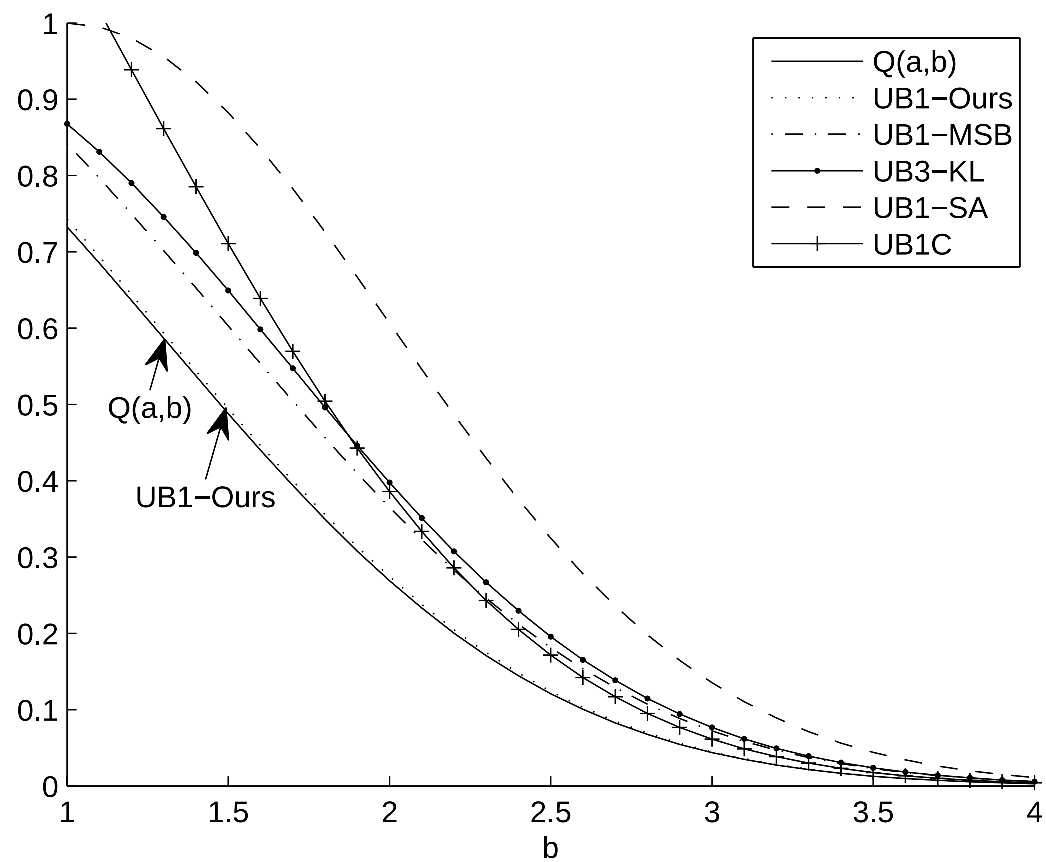

Fig. 9. The upper bounds comparison in the case $a = 1$ and $b \geq a$.

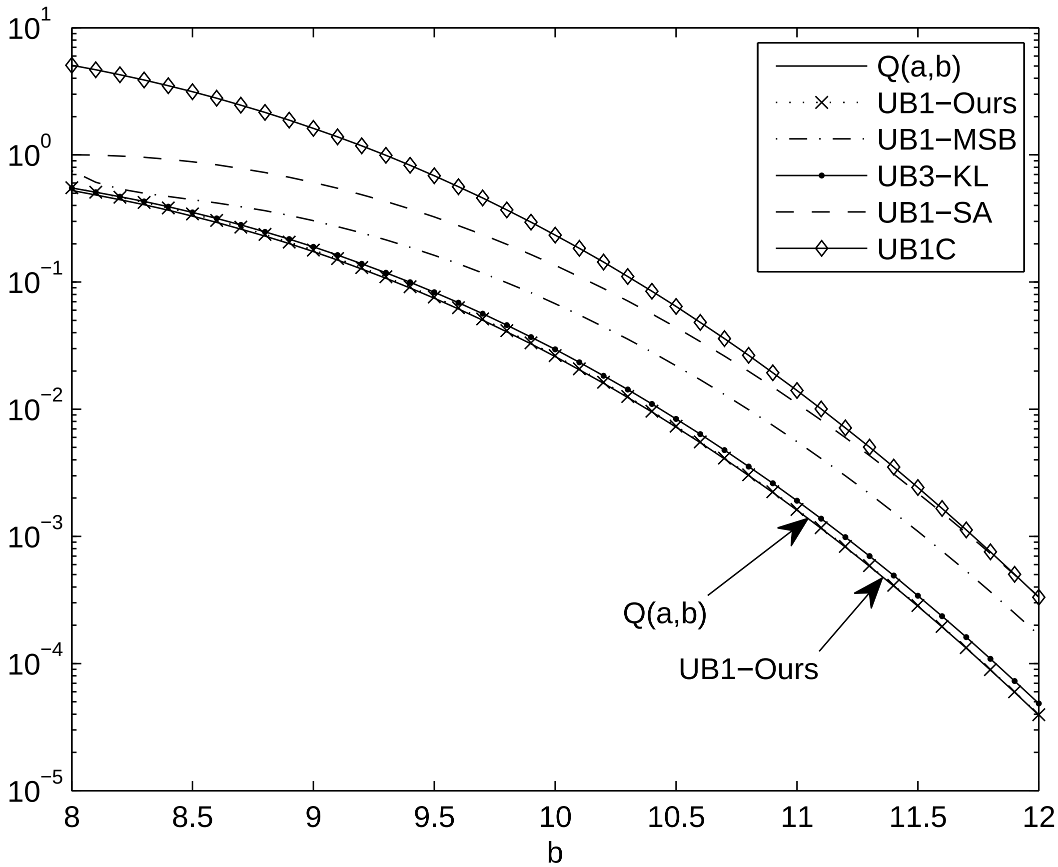

Fig. 10. The upper bounds comparison (logarithmic scale) in the case $a = 8$ and $b \geq a$.

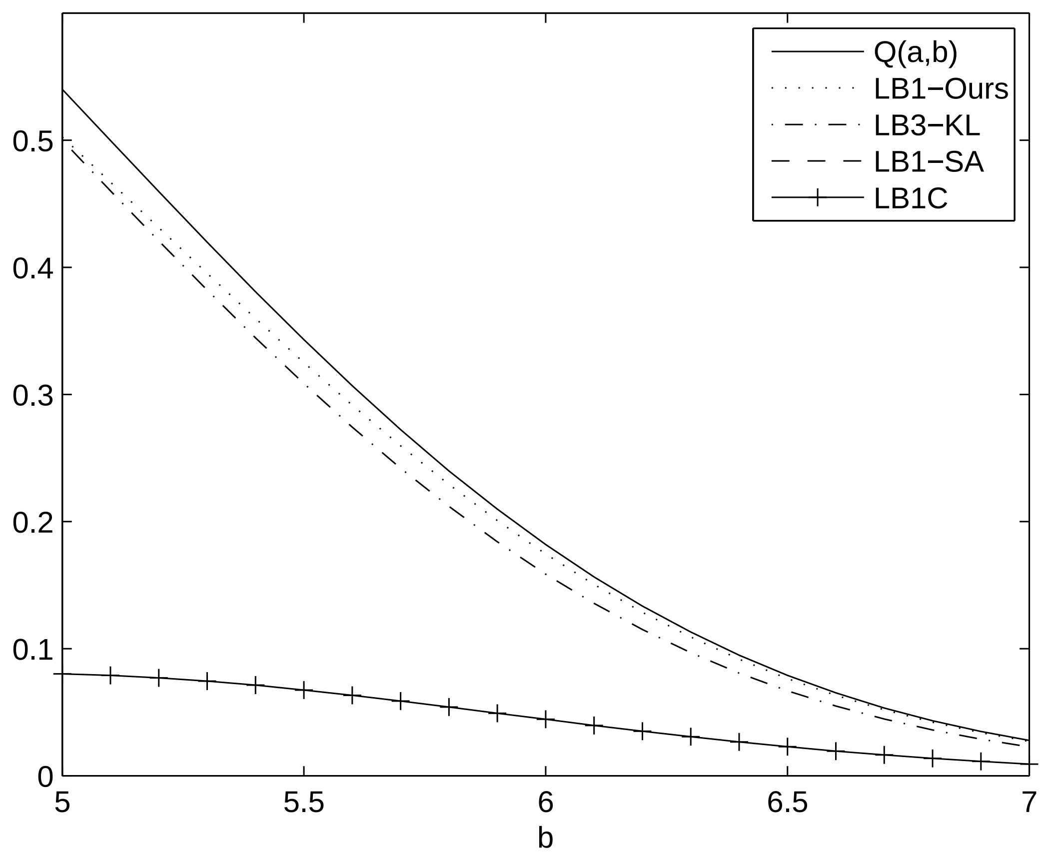

Fig. 11. The lower bounds comparison in the case $a = 5$ and $b \geq a$.

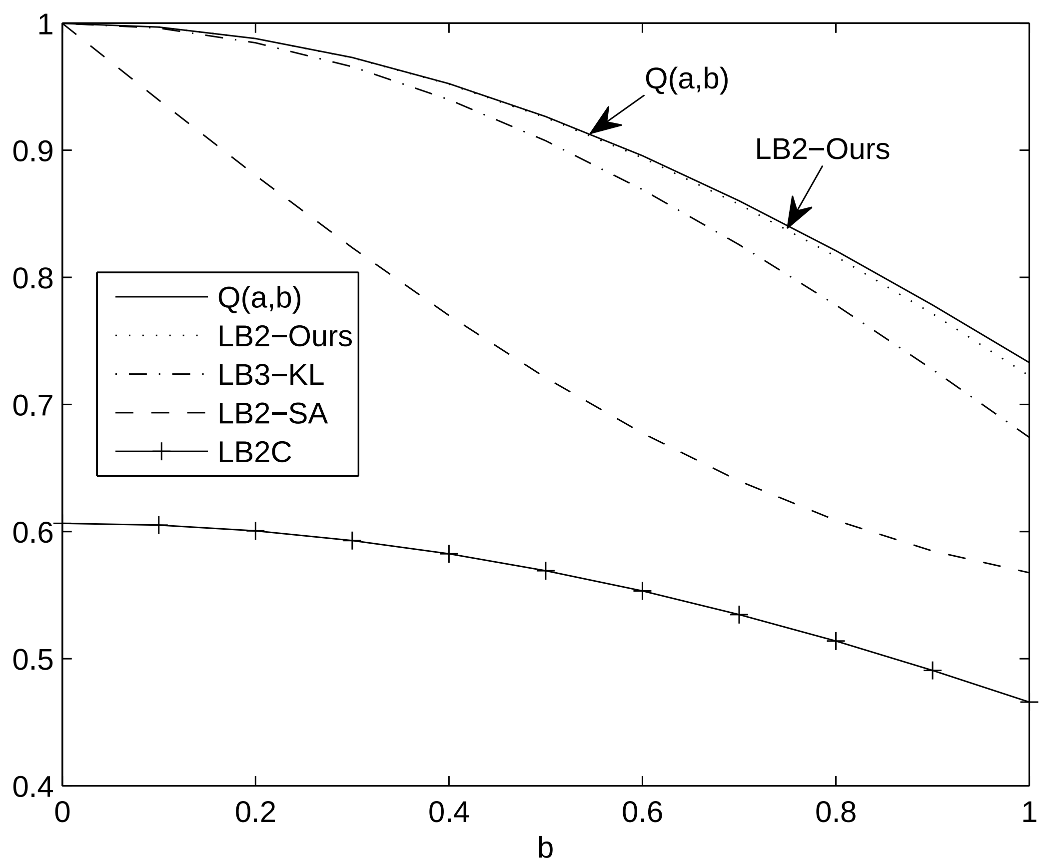

Fig. 12. The lower bounds comparison in the case $a = 1$ and $b \leq a$.

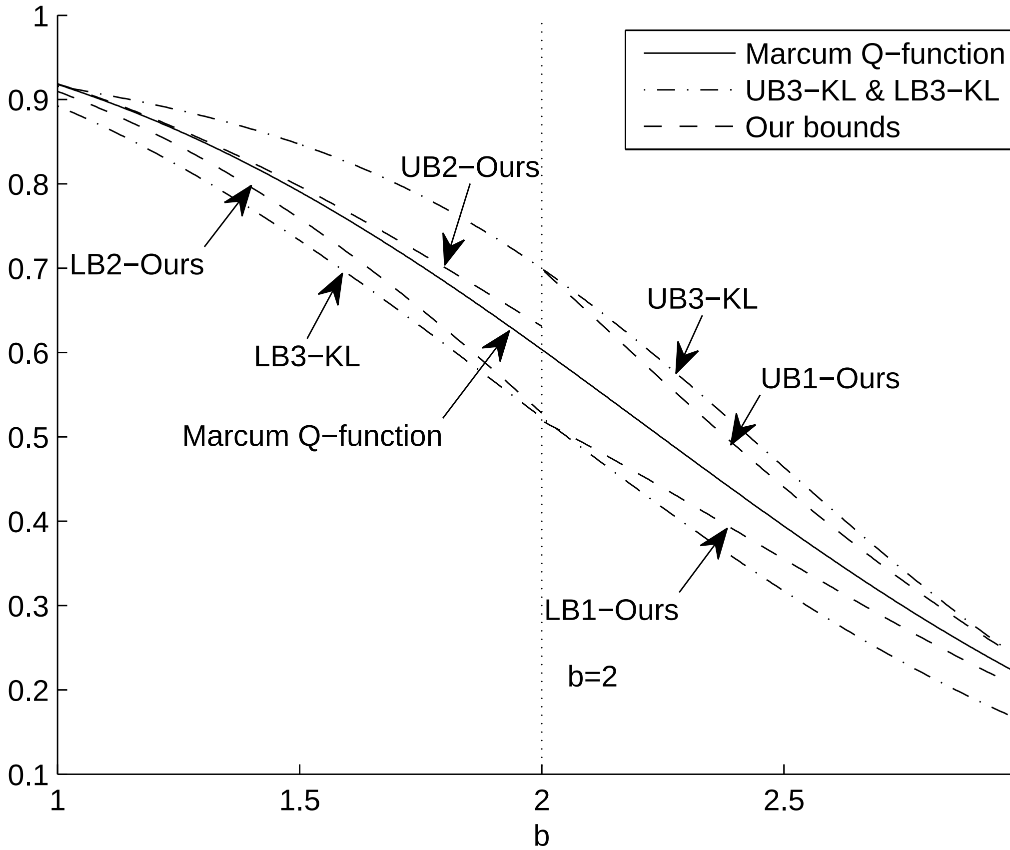

Fig. 13. The comparison between our bounds and KL bounds in [7], in the case $a = 2$.