Example: RVs Marginally Gaussian but not Jointly Gaussian

We have seen that the MMSE estimator takes on a particularly simple form when **x** and **θ** are jointly Gaussian and we went to great lengths to show that this is satisfied for the Bayesian linear model.

The definition of jointly Gaussian is: Two Gaussian RVs X and Y are jointly Gaussian if their joint PDF is a 2-D Gaussian PDF. (Of course, there is an obvious extension to random vectors).

Note the main ingredients here: both RV's must individually be Gaussian and they must have a joint PDF that is Gaussian. This raises the obvious question: Is it possible to have two RVs that are each individually Gaussian but are NOT jointly Gaussian?

The answer is: Yes…. otherwise we wouldn't make such a big stink about this. So let's see if we can find one such case to demonstrate that we DO have to worry about this.

Remember that given a joint PDF $p_{XY}(x,y)$ the individual PDFs are the marginal PDFs that are found by integrating out "the other variable," that is:

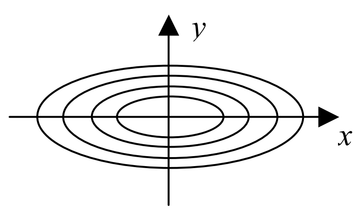
$$p_X(x) = \int p_{XY}(x,y)dy$$

$$p_Y(y) = \int p_{XY}(x,y)dx$$

So we see what we need for our counterexample: we need a joint PDF that is NOT a 2-D Gaussian but that integrates to two Gaussian marginal PDFs. So let's construct one of these. Let's start with a 2-D joint Gaussian PDF and modify it. Define the 2-D Gaussian PDF with zero-mean, uncorrelated RVs, which is then given by:

$$p_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{\frac{-1}{2}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}\right)\right\}$$

which looks like this in a contour plot:

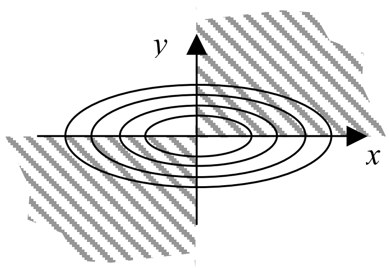


Now, from what we have studied about 2-D Gaussian PDFs, integrating over x this gives a Gaussian marginal in y; likewise, integrating over y gives a Gaussian marginal in x. But because of the symmetry of this joint PDF about both x and y axes we can write these integrations as

$$p_X(x) = \begin{cases} 2\int_0^{\infty} p_{XY}(x,y)dy, & x > 0 \\ 2\int_{-\infty}^{0} p_{XY}(x,y)dy, & x \le 0 \end{cases}$$

$$p_Y(y) = \begin{cases} 2\int_0^{\infty} p_{XY}(x,y)dx, & y > 0 \\ 2\int_{-\infty}^{0} p_{XY}(x,y)dx, & y \le 0 \end{cases}$$

In other words we only have to integrate over the following hatched quadrants to get the marginals, as long as we multiply by 2:



This gives us the route to what we need. If we take this original 2-D Gaussian PDF and set it to zero over the non-hatched quadrants above (the parts we didn't need to create the marginals) and multiply the rest by two we get a new 2-D PDF that is definitely NOT Gaussian:

$$p_{\tilde{X}\tilde{Y}}(x,y) = \begin{cases} \dfrac{1}{\pi\sigma_X\sigma_Y}\exp\left\{\dfrac{-1}{2}\left(\dfrac{x^2}{\sigma_X^2}+\dfrac{y^2}{\sigma_Y^2}\right)\right\}, & \text{when } xy > 0 \\ 0, & \text{when } xy \le 0 \end{cases}$$

The new RVs $\tilde{X}$ and $\tilde{Y}$ are definitely NOT **jointly** Gaussian but they are each Gaussian because (as we have constructed above) the marginals of their joint PDF are Gaussian!